Journal of Statistical Physics, Vol. 43, Nos. 1/2, 1986

Uniqueness of Continuum One-Dimensional Gibbs States for Slowly Decaying Interactions

David Klein[1]

Received March 19, 1985; revised July 9, 1985

We consider one-dimensional grand-canonical continuum Gibbs states corresponding to slowly decaying, superstable, many-body interactions. Absence of phase transitions, in the sense of uniqueness of the tempered Gibbs state, is proved for interactions with an Nth body hardcore for arbitrarily large N.

KEY WORDS: Gibbs state; pure phase; one-dimensional continuum; long range interaction.

1. INTRODUCTION

The presence or absence of phase transitions has been studied for a wide variety of one-dimensional classical statistical mechanical models. The majority of papers on this subject have focused on lattice or hard-core systems, due to the technical difficulties which arise in continuum models without hard-core restrictions (see, however, Campanino et al.,[1] Suhov,[15] Klein[10]). The principal condition on the interaction for the absence of phase transitions, in lattice and hard-core models, is that the total interaction energy of particles distributed along the negative real axis with particles distributed along the positive axis must be finite (see, for example, Dobrushin,[2] Gallavotti et al.,[6] Gallavotti et al.,[7] and Ruelle[12]). In the case of lattice models, Dyson[3,4] and Frohlich et al.[5] have shown that this condition cannot be substantially weakened. Further insight into this question has also been obtained by Simon.[14]

In this paper we study perturbations $V+\Phi_N$ of superstable, slowly decaying, many-body interactions V, where $\Phi_N=\infty$ for configurations with more than N particles in any interval of length one, and $\Phi_N=0$

[1] Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803.

0022-4715/86/0400-0207$05.00/0

822/43/1-2-14

otherwise. If $N=2$, $V+\Phi_N$ is a hard-core interaction in the usual sense. If N is extremely large, so that, for example, N particles in any interval of length one corresponds to a density which greatly exceeds that of any known form of matter, one would not expect physically significant differences in the behavior of systems governed, respectively, by V and $V+\Phi_N$. We prove in Section 3 that if a condition, analogous to that imposed on lattice and hard-core models, holds for V, then the tempered Gibbs state for $V+\Phi_N$ is unique at all temperatures and fugacities, independent of N. The condition is, roughly speaking, that for any given maximum uniform density of particles on the line, the energy of interaction of particles on the negative real line with particles on the positive real line must be finite (see Condition 2.2 and Remark 2.1 below).

This extends results of Dobrushin[2] and the author.[10] The method of proof is based on the ideas of Dobrushin given in Ref. 2.

2. NOTATION AND DEFINITIONS

For a bounded Borel set Λ of the real line, let $X(\Lambda)$ be the set of all locally finite subsets (configurations) of Λ. B_Λ denotes the σ field on $X(\Lambda)$ generated by all sets of the form $\{s\in X(\Lambda): |s\cap B|=m\}$, where B runs over all bounded Borel subsets of Λ, m runs over the set of nonnegative integers, and $|\cdot|$ denotes cardinality. Let X_F be the set of configurations in $X(\mathbb{R})$ of finite cardinality, and $X_N(\Lambda)$ the set of configurations in $X(\Lambda)$ of cardinality N.

Let $T: \Lambda^N \to X_N(\Lambda)$ be the map which takes the ordered N-tuple $(x_1,..., x_N)$ to the unordered set $\{x_1,..., x_N\}$. For $N=1, 2, 3,...$ let $d^N x$ be the projection of n-dimensional Lebesgue measure onto $X_N(\Lambda)$ under the map T. The measure $d^0 x$ assigns mass 1 to $X_0(\Lambda)=\{\emptyset\}$. Define as in Refs. 10, 11,

$$\nu_\Lambda(dx)=\sum_{n=0}^{\infty}\frac{z^n}{n!}d^n x \tag{2.1}$$

where z is chemical activity.

We will consider $B_{\mathbb{R}}$-measurable many-body interactions $V: X_F\to (-\infty, -\infty]$ of the form

$$V(x)=\sum_{N=1}^{\infty}\sum_{\substack{y\subset x\\ |y|=N}}\phi_N(y) \tag{2.2}$$

where $\phi_N: X_N(\mathbb{R})\to(-\infty, +\infty]$ is called an N-body interaction. As in Preston[11] we define the $B_{\mathbb{R}}$-measurable set $R_\Lambda\subset X(\mathbb{R})$ so that $V(x\,|\,s)$

represents the energy of the configuration $x \in X(\Lambda)$, assuming the configuration $s \in R_\Lambda \cap X(\Lambda^c)$. The finite volume Gibbs state $\mu_\Lambda(dx \mid s)$ for the bounded Borel set Λ (with positive Lebesgue measure), interaction V, inverse temperature β, chemical activity z, and external configuration $s \in R_\Lambda \cap X(\Lambda^c)$ is given by

$$\mu_\Lambda(dx \mid s) = \frac{\exp[-\beta V(x \mid s)]}{Z_\Lambda(s)} \, \nu_\Lambda(dx) \tag{2.3}$$

where the constant $Z_\Lambda(s)$ makes $\mu_\Lambda(dx \mid s)$ a probability measure. If V satisfies Condition 2.1 below, then $1 \leqslant Z_\Lambda(s) < \infty$. If $s \notin R_\Lambda$, define $\mu_\Lambda(dx \mid s)$ to be the zero measure.

Definition 2.1. For a given interaction V, let $D = \{s \in X(\mathbb{R})$: $V(y) < \infty$ for all $y \subset s$ with $|y| < \infty\}$ and let

$$U_m = \{s \in X(\mathbb{R}) \colon |s \cap [-L, L)| < 2Lm \text{ for every integer } L > 0\} \cap D$$

Let $U_\infty = \bigcup_{m \geqslant 1} U_m$.

We will refer to the following two conditions on the interaction V, in the next section.

Condition 2.1. (a) V is superstable.[13]

(b) For any bounded Borel set $\Lambda \subset \mathbb{R}$, any $x \in X(\Lambda) \cap U_\infty$, and any $m \geqslant 1$, $|V_\Lambda(x \mid s) - V_\Lambda(x \mid s \cap [-k, k])| \leqslant \varepsilon_m(k)\,|x|$, where $\varepsilon_m(k)$ converges uniformly to zero for all $s \in U_m \cap X(\Lambda^c)$, as $k \to \infty$.

(c)

$$\sum_{N \geqslant 2} \sum_{\substack{y \subset x \cup s \\ |y| = N \\ y \cap s \neq \emptyset \\ y \cap x \neq \emptyset}} \phi_N(y) \geqslant -c\,|x|\,|s|$$

for some $c > 0$ and all disjoint $x, s \in X_F$.

In Condition 2.2 below, for $y \in X(\mathbb{R})$, let $y^+ = y \cap [0, \infty)$ and $y^- = y \cap (-\infty, 0)$.

Condition 2.2. (a) ϕ_N is translation invariant for each $N \geqslant 1$.

(b) There exists a decreasing function $\psi_m \colon [0, \infty) \to [0, \infty)$, depending on m, such that $\psi_m(r) \to 0$ as $r \to \infty$, and for any $x \in U_m$,

$$\left| \sum_{N \geqslant 2} \sum_{y \in M_x^N(r)} \phi_N(y) \right| \leqslant \psi_m(r)$$

where $M_x^N(r) = \{y \subset x \colon |y| = N,\ \ y^+ \neq \emptyset,\ \ y^- \neq \emptyset$ and $\max_{y_i \in y^+, y_j \in y^-} |y_i - y_j| \geqslant r\}$.

Definition 2.2. For a configuration $x=(x_1,..., x_N)\in X_N(\mathbb{R})$, let

$$\varphi_N(x)=\begin{cases}\infty & \text{if } \max_{ij} |x_i-x_j|<1\\ 0 & \text{otherwise}\end{cases}$$

For a given interaction V, let

$$V^N(x)=V(x)+\sum_{\substack{y\subset x\\ |y|=N}} \varphi_N(y) \tag{2.4}$$

We also let U_m^N and U_∞^N (from Definition 2.1) and $Z_A^N(s)$ from (2.3) correspond to V^N.

Remark 2.1. Condition 2.1 was required in Ref. 9 for existence of Gibbs states for V^N. Condition 2.2(a) is not essential, but without it Condition 2.2(b) would be harder to state. Condition 2.2(b) can be understood roughly in the following way: For any prescribed uniform density and any configuration in $\mathbb{R}$ not exceeding that density, the energy of interaction of particles in $(-\infty, 0)$ with those in $[0, \infty)$ is finite. In the case of a pair interaction $V(x)=\sum_{ij}\phi_2(|x_i-x_j|)$ for which

$$|\phi_2(|x|)|\leqslant C\,|x|^{-\alpha}$$

when $|x|$ is sufficiently large, V satisfies Condition 2.2 for any $\alpha>2$. Condition 2.1 is also satisfied by V, if in addition, V is superstable and $\inf_{x>0}\phi_2(x)>-\infty$.

Remark 2.2. If V satisfies Condition 2.1 and Condition 2.2, then V^N also satisfies Condition 2.1 and Condition 2.2. Condition 2.2(b) can be simplified for V^N by replacing ψ_m with ψ_N, since $U_N^N=U_\infty^N\supset U_m^N$ for all $m\geqslant 1$.

Let $\{\pi_A\}$ denote the specification associated with β, z, and V (see Ref. 11, p. 16) defined by

$$\pi_A(A, s)=\int_{A'}\mu_A(dx\,|\,s\cap V^c) \tag{2.5}$$

where $A\in B_{\mathbb{R}}$, $A'=\{x\in X(A)\colon x\cup(s\cap A^c)\in A\}$, and $s\in X(\mathbb{R})$.

Definition 2.3. A probability measure σ on $(X(\mathbb{R}), B_{\mathbb{R}})$ is a Gibbs state for the specification $\{\pi_A\}$ if

$$\sigma[\pi_A(A, s)]=\sigma(A)$$

for every $A\in B_{\mathbb{R}}$ and every bounded Borel set $A\subset\mathbb{R}$ of positive Lebesgue measure. If in addition, $\sigma(U_\infty)=1$, then σ is a tempered Gibbs state.

Definition 2.4. For an interaction V, Borel sets $\Lambda \subset \tilde{\Lambda}$ with positive finite Lebesgue measures, and $s \in U_\infty$, the finite volume Gibbs density $r_{\tilde{\Lambda}}^{\Lambda}(x \mid s)$ is given by

$$r_{\tilde{\Lambda}}^{\Lambda}(x \mid s) = \int_{X(\tilde{\Lambda} \setminus \Lambda)} \frac{\exp[-\beta V(x \cup y \mid s \cap \tilde{\Lambda}^c)]}{Z_{\tilde{\Lambda}}(s \cap \tilde{\Lambda}^c)} \, \nu_{\tilde{\Lambda} \setminus \Lambda}(dy) \tag{2.6}$$

Definition 2.5. A function f on $X(\mathbb{R})$ is a cylinder function if $f(s) = f(s \cap \Lambda)$ for some bounded set $\Lambda \subset \mathbb{R}$ and all $s \in X(\mathbb{R})$. A subset $A \subset X(\mathbb{R})$ is a cylinder set if the characteristic function for A is a cylinder function.

Note that if f is a B_Λ-measurable function on $X(\Lambda)$, then we may regard f as a $B_{\mathbb{R}}$-measurable cylinder function on $X(\mathbb{R})$ by defining $f(s) = f(s \cap \Lambda)$ for $s \in X(\mathbb{R})$. In this case

$$\pi_{\tilde{\Lambda}}(f, s) \equiv \int f(x) \, \mu_{\tilde{\Lambda}}(dx \mid s) = \int_{X(\Lambda)} f(x) \, r_{\tilde{\Lambda}}^{\Lambda}(x \mid s) \, \nu_\Lambda(dx) \tag{2.7}$$

3. UNIQUENESS OF THE GIBBS STATE

We assume throughout that an interaction V is given which satisfies Condition 2.1 and Condition 2.2 of the last section.

Let (X, B_X) be a measurable space and let μ_1 and μ_2 be probability measures on (X, B_X). The variation distance between the measures μ_1 and μ_2 is defined as

$$\rho(\mu_1, \mu_2) = \sup_{A \in B_X} |\mu_1(A) - \mu_2(A)| \tag{3.1}$$

If μ_1 and μ_2 have respective densities p_1 and p_2 with respect to a finite measure ν on X, then defining $\rho(p_1, p_2) = \rho(\mu_1, \mu_2)$, we have

$$\rho(p_1 \, p_2) = \tfrac{1}{2} \int_X |p_1(x) - p_2(x)| \, \nu(dx) = 1 - \int_X \min[p_1(x), p_2(x)] \, \nu(dx) \tag{3.2}$$

The following lemma was proved by Dobrushin.[2]

Lemma 3.1. (Dobrushin). Let (X_j, B_j, ν_j) be a measure space for $j = 1, 2, 3$ and let

$$(X, B_X, \nu) = \prod_{j=1}^{3} (X_j, B_j, \nu_j)$$

be the product measure space with measure $\nu = \nu_1 \times \nu_2 \times \nu_3$. Let $p^1(\cdot)$ and

$p^2(\cdot)$ be densities with respect to ν for probability measures on (X, B_X). Consider the marginal densities

$$p_1^i(x_1) = \iint p^i(x_1, x_2, x_3)\, \nu_2(dx_2)\, \nu_3(dx_3)$$

$$p_{1,2}^i(x_1, x_2) = \int p^i(x_1, x_2, x_3)\, \nu_3(dx_3) \qquad \text{for} \quad i = 1, 2$$

and the similarly defined densities $p_2^i(x_2)$, $p_3^i(x_3)$, $p_{1,3}^i(x_1, x_3)$, and $p_{2,3}^i(x_2, x_3)$ for $i = 1, 2$. Suppose there exist conditional densities $p_1^i(x_1 \mid x_2, x_3)$ and $p_{1/2}^i(x_1 \mid x_2)$ for which

$$p^i(x_1, x_2, x_3) = p_1^i(x_1 \mid x_2, x_3)\, p_{2,3}^i(x_2, x_3)$$
$$p_{1,2}^i(x_1, x_2) = p_{1/2}^i(x_1 \mid x_2)\, p_2^i(x_2) \quad (i = 1, 2)$$

Then

$$\rho(p_1^1, p_1^2) \leqslant \alpha_0 \rho(p_2^1, p_2^2) + \bar{\alpha}_0[1 - \rho(p_2^1, p_2^2)]$$

where

$$\alpha_0 = \sup_{\substack{x_j, \tilde{x}_j \in X_j \\ j = 1,2}} \rho(p_1^1[(\cdot \mid x_2, x_3), p_1^2(\cdot \mid \tilde{x}_2, \tilde{x}_3)]$$

$$\bar{\alpha}_0 = \sup_{\substack{x_2 \in X_2 \\ x_3, \tilde{x}_3 \in X_3}} \rho[p_1^1(\cdot \mid x_2, x_3), p_1^2(\cdot \mid x_2, \tilde{x}_3)]$$

As in Ref. 2, uniqueness of the Gibbs state σ_N for the interaction V^N given in Definition 2.2 holds provided

$$\lim_{n \to \infty} \sup_{s,t \in U_\infty^N} \rho[r_{[-n,n]}^I(\cdot \mid s), r_{[-n,n]}^I(\cdot \mid t)] = 0 \tag{3.3}$$

for all sufficiently large finite intervals $I \subset \mathbb{R}$, where $r_{[-n,n]}^I(\cdot \mid \cdot)$ corresponds to V^N via Definition 2.4.

We will use Lemma 2.1 to establish (3.3).

Let an interval $(a, c]$ be given and let $s_1, s_2 \in U_\infty^N$ satisfy $s_1 \cap (c, \infty) = s_2 \cap (c, \infty)$. Assume n is large enough so that $[-n, n] \supset (a, c]$, and define $b = (a + c)/2$. In the language of Dobrushin's lemma we make the following identifications

$$\begin{aligned} (X_1, B_1) &= [X((b, c]), B_{(b,c]}] \\ (X_2, B_2) &= [X((a, b]), B_{(a,b]}] \\ (X_3, B_3) &= [X([-n, a]), B_{[-n,a]}] \end{aligned} \tag{3.4}$$

Given a configuration $x \in X(\mathbb{R})$, let

$$x_1 = x \cap (b, c], \qquad x_2 = x \cap (a, b], \qquad x_3 = x \cap [-n, a]$$

and

$$\nu = \nu_{[-n,a]} \times \nu_{(a,b]} \times \nu_{(b,c]}$$

For $i = 1, 2$, let

$$p^i(x_1, x_2, x_3) = r^{[-n,c]}_{[-n,n]}(x_1, x_2, x_3 \mid s_i) \tag{3.5}$$

It follows as in Ref. 2 that for $i = 1, 2$

$$p^i_1(x_1) = r^{(b,c]}_{[-n,n]}(x_1 \mid s_i) \tag{3.6}$$

$$p^i_2(x_2) = r^{(a,b]}_{[-n,n]}(x_2 \mid s_i) \tag{3.7}$$

$$p^i_1(x_1 \mid x_2, x_3) = r^{(b,c]}_{(b,n]}[x_1 \mid x_2 \cup x_3 \cup (s_i \backslash [-n, b])] \tag{3.8}$$

Remark 2.1. For the probability densities just defined, the conclusion of Lemma 3.1 can be modified, with no change in Dobrushin's proof, so that

$$\rho(p^1_1, p^2_1) \leqslant \alpha \rho(p^1_2, p^2_2) + \bar{\alpha}[1 - \rho(p^1_2, p^2_2)] \tag{3.9}$$

where

$$\begin{aligned} \alpha = \sup\{&\rho[p^1_1(\cdot \mid x_2, x_3), p^2_1(\cdot \mid \tilde{x}_2, \tilde{x}_3)]: \\ &x_2 \cup x_3 \in U^N_\infty, \tilde{x}_2 \cup \tilde{x}_3 \in U^N_\infty, x_j, \tilde{x}_j \in X_J \qquad \text{for} \quad j = 1, 2\} \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \bar{\alpha} = \sup\{&\rho[p^1_1(\cdot \mid x_2, x_3), p^2_1(\cdot \mid x_2, \tilde{x}_3)]: x_2 \in X_2, x_3, \tilde{x}_3 \in X_3 \\ &\text{and} \qquad x_2 \cup x_3 \in U^N_\infty, x_2 \cup \tilde{x}_3 \in U^N_\infty\} \end{aligned} \tag{3.11}$$

Lemma 3.2. For any integer $N \geqslant 2$, the following inequality holds for the interaction V^N

$$\bar{\alpha} \leqslant \beta \psi_N(c - b)$$

where $\psi_N(\cdot)$ is given by Condition 2.2 and $\bar{\alpha}$ is given by (3.11).

Proof. Since the distribution with density $r^{(b,c]}_{(b,n]}(\cdot \mid t_i)$ is the restriction of the distribution with density $r^{(b,n]}_{(b,n]}(\cdot \mid t_i)$ onto a smaller σ algebra, it follows that

$$\rho[r^{(b,c]}_{(b,n]}(\cdot \mid t_1), r^{(b,c]}_{(b,n]}(\cdot \mid t_2)] \leqslant \rho[r^{(b,n]}_{(b,n]}(\cdot \mid t_1), r^{(b,n]}_{(b,n]}(\cdot \mid t_2)] \tag{3.12}$$

for any $t_1, t_2 \in U_\infty^N$. Combining (3.2) and (3.11) with (3.12) gives

$$\bar{\alpha} \leqslant \tfrac{1}{2} \sup \int_{X((b,n])} \left| \frac{\exp[-\beta V^N(y \mid x_2 \cup t_1)]}{Z_{(b,n]}^N(x_2 \cup t_1)} - \frac{\exp[-\beta V^N(y \mid x_2 \cup t_2)]}{Z_{(b,n]}^N(x_2 \cup t_2)} \right| \nu_{(b,n]}(dy) \tag{3.13}$$

where the supremum is taken over the same set of configurations as in (3.14) below. From Lemma 3.3 of Ref. 8, the right side of (3.13) is bounded by

$$\tfrac{1}{2}\beta \sup\{|V^N(y \mid x_2 \cup t_1) - V^N(y \mid x_2 \cup t_2)| : t_i \cup x_2 \cup y \in U_\infty^N$$
$$t_i \cap (a, n] = \emptyset \quad \text{for} \quad i = 1, 2, \quad \text{and} \quad t_1 \cap (n, \infty) = t_2 \cap (n, \infty)\} \tag{3.14}$$

By Condition 2.2 and Remark 2.2, the above expression is bounded by $\beta\psi_N(b-a) = \beta\psi_N(c-b)$. This concludes the proof.

Lemma 3.3. For any $N \geqslant 2$, $n \geqslant b+1$, there exists $h_1 > 0$ such that

$$\frac{\exp[-\beta V^N(\emptyset_J \cup y \mid s_2)]}{Z_{(b,n]}^N(s_2)} \geqslant h_1 \frac{\exp[-\beta V^N(\emptyset_J \cup y \mid s_1)]}{Z_{(b,n]}^N(s_1)} \tag{3.15}$$

where $J = (b, b+1]$, $\emptyset_J$ is the empty configuration in J, and $y \in X((b+1, n]) \cap U_\infty^N$ is arbitrary. Furthermore, h_1 depends only on β, z, and N.

Proof. From Lemma 2.1 of Ref. 9, for any $y_1 \in X(J) \cap U_\infty^N$, there exists a constant $D > 0$ depending only on J and N such that if $y \cup s_i \in U_\infty^N$,

$$V^N(y_1 \mid s_i \cup y) \geqslant -D\,|y_1| \geqslant -DN \tag{3.16}$$

Since $V^N(y_1 \cup y \mid s_2) = V^N(\emptyset_J \cup y \mid s_2) + V^N(y_1 \mid y \cup s_2)$, we get

$$V^N(\emptyset_J \cup y \mid s_2) - V^N(y_1 \cup y \mid s_2) \leqslant DN \tag{3.17}$$

From (3.17) it follows that

$$\int_{X(J)} \int_{X((b+1,n])} \exp[-\beta V^N(\emptyset_J \cup y \mid s_2)]\, \nu_J(dy_1)\, \nu_{(b+1,n]}(dy)$$
$$\geqslant \exp[-\beta DN] \int_{X(J)} \int_{X((b+1,n])} \exp[-\beta V^N(y_1 \cup y \mid s_2)]\, \nu_J(dy_1)\, \nu_{(b+1,n]}(dy) \tag{3.18}$$

Since $\nu_{(b,n]} = \nu_J \times \nu_{(b+1,n]}$, we have

$$\begin{aligned}\int_{X((b+1,n])} \exp[-\beta V^N(\emptyset_J \cup y \mid s_2)]\, \nu_{(b+1,n]}(dy) \\ &\geqslant \exp[-\beta DN](\nu_J[X(J)])^{-1} Z^N_{(b,n]}(s_2) \\ &= \exp[-\beta DN - z]\, Z^N_{(b,n]}(s_2) \qquad (3.19)\end{aligned}$$

From Condition 2.2 we also have

$$|V^N(\emptyset_J \cup y \mid s_2) - V^N(\emptyset_J \cup y \mid s_1)| \leqslant 2\psi_N(1)$$

and consequently

$$\exp[-\beta V^N(\emptyset_J \cup y \mid s_2)] \geqslant \exp[-2\beta\psi_N(1)] \exp[-\beta V^N(\emptyset_J \cup y \mid s_1)] \qquad (3.20)$$

Combining (3.19) and (3.20) gives

$$\begin{aligned}Z^N_{(b,n]}(s_1) &\geqslant \int_{X((b+1,n])} \exp[-\beta V^N(\emptyset_J \cup y \mid s_1)]\, \nu_{(b+1,n]}(dy) \\ &\geqslant \exp[-2\beta\psi_N(1)] \int_{X((b+1,n])} \exp[-\beta V^N(\emptyset_J \cup y \mid s_2)]\, \nu_{(b+1,n]}(dy) \\ &\geqslant \exp[-\beta DN - 2\beta\psi_N(1) - z]\, Z^N_{(b,n]}(s_2) \qquad (3.21)\end{aligned}$$

Combining (3.20) with (3.21) then gives

$$\begin{aligned}\frac{\exp[-\beta V^N(\emptyset_J \cup y \mid s_2)]}{Z^N_{(b,n]}(s_2)} \\ &\geqslant \exp[-\beta DN - 4\beta\psi_N(1) - z] \frac{\exp[-\beta V^N(\emptyset_J \cup y \mid s_1)]}{Z^N_{(b,n]}(s_1)}\end{aligned}$$

This completes the proof.

Lemma 3.4. Assume $n \geqslant c \geqslant b+1$. For any $\beta, z > 0$ and $N \geqslant 2$, there exists $h > 0$ independent of n, c, b, s_1, and s_2 such that

$$\rho[r^{(b,c]}_{(b,n]}(\cdot \mid s_1), r^{(b,c]}_{(b,n]}(\cdot \mid s_2)] \leqslant 1 - h \qquad (3.22)$$

Proof. From (3.16) it follows that

$$\sup_{\substack{s \in U^N_\infty \\ s \cap J = \emptyset}} \int_{X(J)} \exp[-\beta V^N_J(x \mid s)]\, \nu_J(dx) < \infty \qquad (3.23)$$

Since $\exp[-\beta V^N(\emptyset_J \mid t)] = \exp 0 = 1$ for any t,

$$\inf_{t \in U_\infty^N} \pi_J^N(\{\emptyset_J\}, t) \equiv h_2 > 0 \tag{3.24}$$

where we have used the same notation as Lemma 3.3. From the consistency of the specification $\{\pi_\Lambda^N\}$ (see Ref. 11), we have

$$\pi_{(b,n]}^N(\{\emptyset_J\}, s) = \int \pi_J^N(\{\emptyset_J\}, t)\, \pi_{(b,n]}^N(dt, s) \geqslant h_2 \tag{3.25}$$

From (2.6) and (2.7) it follows that

$$\pi_{(b,n]}^N(\{\emptyset_J\}, s) = \int_{\{\emptyset_J\} \cup X((b+1,n])} r_{(b,n]}^{(b,n]}(y \mid s)\, v_{(b,n]}(dy) \geqslant h_2 \tag{3.26}$$

where h_2 is independent of b, n, and s. From (3.15)

$$\int_{\{\emptyset_J\} \cup X((b+1,n])} \min[r_{(b,n]}^{(b,n]}(y \mid s_1), r_{(b,n]}^{(b,n]}(y \mid s_2)]\, v_{(b,n]}(dy) \geqslant h_1 h_2 \equiv h \tag{3.27}$$

Now combining (3.2), (3.12), and (3.27) gives the desired result.

Corollary 3.1. With the same assumptions as in Lemma 3.4,

(a) $\alpha \leqslant 1-h$ for some $h>0$ depending only on V^N, β, z (and not s_1, s_2, b, c, or n).

(b) $$\rho(r_{[-n,n]}^{(b,c]}(\cdot \mid s_1), r_{[-n,n]}^{(b,c]}(\cdot \mid s_2)) \leqslant (1-h)\,\rho(r_{[-n,n]}^{(a,b]}(\cdot \mid s_1), r_{[-n,n]}^{(a,b]}(\cdot \mid s_2)) + \beta\psi_N(c-b),$$

where $\psi_N(\cdot)$ is given by Condition 2.2.

Proof. Part (a) follows from (3.10) and Lemma 3.4. Part (b) follows from (3.9), with identifications given by (3.4) to (3.8), and from Lemma 3.2.

Theorem 3.1. For any $\beta, z>0$, any interaction V satisfying Condition 2.1 and Condition 2.2, and any integer $N \geqslant 2$, there is exactly one tempered Gibbs state for V^N, β, z.

Proof. A simple induction argument together with Corollary 3.1(b) shows that given any positive integer m, it is possible to choose n large enough so that

$$\rho(r_{[-n,n]}^{(b,c]}(\cdot \mid s_1), r_{[-n,n]}^{(b,c]}(\cdot \mid s_2)) \leqslant (1-h)^{m+1} + m\beta\psi_N(c-b) \tag{3.28}$$

as long as $c \geqslant b+1$. Thus given any $\varepsilon>0$, it is possible to choose m large enough so that $(1-h)^{m+1} < \varepsilon/2$, and then to choose $c-b$ large enough so that $m\beta\psi_N(c-b) < \varepsilon/2$ and finally to choose n large enough so that (3.28) holds for these choices of m and $c-b$. Hence for any sufficiently large interval $(b, c]$ and sufficiently large n

$$\rho(r^{(b,c]}_{[-n,n]}(\cdot \mid s_1), r^{(b,c]}_{[-n,n]}(\cdot \mid s_2)) < \varepsilon \tag{3.29}$$

for all $s_1, s_2 \in U^N_\infty$ with $s_1 \cap (c, \infty) = s_2 \cap (c, \infty)$.

Now given any $s, t \in U^N_\infty$, let s_1 be chosen so that $s_1 \cap (c, \infty) = s$ and $s_1 \cap (-\infty, b] = t$. By the triangle inequality

$$\begin{aligned}\rho(r^{(b,c]}_{[-n,n]}(\cdot \mid s), r^{(b,c]}_{[-n,n]}(\cdot \mid t)) &\leqslant \rho(r^{(b,c]}_{[-n,n]}(\cdot \mid s), r^{(b,c]}_{[-n,n]}(\cdot \mid s_1)) \\ &\quad + \rho(r^{(b,c]}_{[-n,n]}(\cdot \mid s_1), r^{(b,c]}_{[-n,n]}(\cdot \mid t)\end{aligned} \tag{3.30}$$

The first term on the side of (3.30) is bounded according to (3.29). Similarly, using a relation proved by obvious analogy with (3.29), we can conclude that the second term on the right side of (3.30) can be made arbitrarily small for sufficiently large $c-b$ and n. It follows that for $(b, c]$ sufficiently large

$$\lim_{n \to \infty} \sup_{s,t \in U^N_\infty} \rho(r^{(b,c]}_{[-n,n]}(\cdot \mid s), r^{(b,c]}_{[-n,n]}(\cdot \mid t)) = 0$$

This guarantees uniqueness of the Gibbs state for V^N, β, z as in Ref. 10. Existence was established in Ref. 9 and it is easy to see that any Gibbs state corresponding to V^N must be tempered. This completes the proof.

Remark 3.1. The results on high temperature decay of correlations given in Ref. 9 holds for the interactions considered here.

Remark 3.2. For an interaction V satisfying Conditions 2.1 and 2.2 without any hard core, and for any fixed finite volume, the finite volume Gibbs states for V^N converge to the finite volume Gibbs states for V for any boundary condition $s \in U^\infty$, as $N \to \infty$. One might therefore expect that uniqueness of the infinite volume tempered Gibbs state for V would follow from the uniqueness of the infinite volume Gibbs state for V^N, for each N. Such an argument was given in Ref. 10. The argument is incorrect due principally to an incorrect definition of a tempered Gibbs state given there.

REFERENCES

1. M. Campanino, D. Capocaccia, and E. Olivieri, *J. Stat. Phys.* **33**:437–476 (1983).
2. R. L. Dobrushin, *Math. Sb.* **93**:29–49 (1974).
3. F. Dyson, *Comm. Math. Phys.* **12**:91–107 (1969).

4. F. Dyson, *Comm. Math. Phys.* **21**:269–283 (1971).
5. J. Frohlich, R. Isreal, E. Lieb, and B. Simon, *Comm. Math. Phys.* **62**:1–34 (1978).
6. G. Gallavotti, S. Miracle-Sole, and D. Ruelle, *Phys. Lett.* **A26**:350–351 (1968).
7. G. Gallavotti and S. Miracle-Sole, *J. Math. Phys.* **11**:147–154 (1969).
8. D. Klein, *Comm. Math. Phys.* **86**:227–246 (1982).
9. D. Klein, *Comm. Math. Phys.* **92**:295–308 (1984).
10. D. Klein, *Comm. Math. Phys.* **97**:595–604 (1985).
11. C. Preston, "Random fields," in *Lecture Notes in Mathematics*, Vol. 534 (Springer, Berlin, Heidelberg, New York, 1976).
12. D. Ruelle, *Comm. Math. Phys.* **9**:267–278 (1968).
13. D. Ruelle, *Comm. Math. Phys.* **18**:127–159 (1970).
14. B. Simon, *J. Stat. Phys.* **26**:307–311 (1981).
15. Yu. Suhov, *Comm. Math. Phys.* **50**:113–132 (1976).